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Maximal Eigenvalue and norm of the product of Toeplitz matrices. Study of a particular case.

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Abstract

Maximal eigenvalue and norm of the product of Toeplitz matrices. Study of a particular case
In this paper we describe the asymptotic behaviour of the spectral norm of the product of two finite Toeplitz matrices as the matrix dimension goes to infinity. These Toeplitz matrices are generated by positive functions with Fisher-Hartwig singularities of negative order. Since we have positive operators it is known that the spectral norm is also the largest eigenvalue of this product.

1 Introduction

If $f \in L^1(\mathbb{T})$ the Toeplitz matrix with symbol f denoted by $T_N(f)$ is the $(N+1)\times(N+1)$ matrix such that

$$(T_N(f))_{i+1,j+1} = \hat{f}(j-i) \quad \forall i,j \quad 0 \le i,j \le N$$

(see, for instance, [6],[7]). We say that a function h is regular if $h \in L^\infty(\mathbb{T})$ and $h > 0$. Otherwise the function h is said singular. If b is a regular function continuous in $e^{i\theta_r}$ we call Fisher-Hartwig symbols the functions

$$f(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} |e^{i\theta} - e^{i\theta_r}|^{2\alpha_r} \varphi_{\beta_r,\theta_r}(e^{i\theta})$$

where

- the complex numbers α_r and β_r are subject to the constraints $-\frac{1}{2} < \alpha_r < \frac{1}{2}$ and $-\frac{1}{2} < \beta_r < \frac{1}{2}$,
- the functions $\varphi_{\beta_r,\theta_r}$ are defined as $\varphi_{\beta_r,\theta_r}(e^{i\theta}) = e^{i\beta_r(\pi+\theta-\theta_r)}$.

The problem of the extreme eigenvalues of a Toeplitz matrix is well known (see [12] and [1]). If $\lambda_{k,N}$ $1 \le k \le N+1$ are the eigenvalues of $T_N(f)$ with $\lambda_{1,N} \le \lambda_{2,N} \cdots \le \lambda_{N+1,N}$ we have

$$\lim_{N\to=+\infty} \lambda_{1,N} = m_f \quad \text{and} \quad \lim_{N\to+\infty} \lambda_{N+1,N} = M_f$$

with $m_f = \operatorname{essinf} f$ and $M_f = \operatorname{essup} f$. In [5] and [8] Böttcher and Grudsky on one hand and Böttcher and Virtanen in the other hand give an asymptotic estimation of the maximal eigenvalue in the case of one Toeplitz matrix when the symbol has one or several zeros of

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negative order. In [13] we have obtained the asymptotic of the minimal eigenvalue of one Toeplitz matrix when the symbol has one zero of order α with $\alpha > \frac{1}{2}$.

But estimatig the eigenvalues of the product of two Toeplitz matrices is more delicate. Effectively it is clear that a product of Toeplitz matrices is generally not a Toeplitz matrix. In the first part of this paper we consider the product $T_N(f_1)T_N(f_2)$ of two Toeplitz matrices where $f_1(e^{i\theta}) = |1-e^{i\theta}|^{-2\alpha_1}c_1(e^{i\theta})$, and $f_2(e^{i\theta}) = |1-e^{i\theta}|^{-2\alpha_2}c_2(e^{i\theta})$ with $0 < \alpha_1, \alpha_2 < \frac{1}{2}$ and c_1, c_2 are two regular continuous functions on the torus. For these symbols we obtain the norm of the matrix $T_N(f_1)T_N(f_2)$. Owing to an important result of Widom (see Lemma 3 and also [18], [17], [16], [9]), which connects the norm of an operator and the norm of a matrix. A proof of this result can be found in [8]. Since $T_N(f_1)T_N(f_2)$ is a positive matrix the norm is also the maximal eigenvalue of this matrix. Hence our main result (see Theorem 3) can be also stated as

Theorem 1 *Let $f_1(e^{i\theta}) = |1-e^{i\theta}|^{-2\alpha_1}c_1(e^{i\theta})$ and $f_2(e^{i\theta}) = |1-e^{i\theta}|^{-2\alpha_2}c_2(e^{i\theta})$ with $0 < \alpha_1, \alpha_2 < \frac{1}{2}$ and $c_1, c_2 \in L^\infty(\mathbb{T})$ continuous and nonzero in 1. Then if $\Lambda_{\alpha_1,\alpha_2,N}$ is the maximal eigenvalue of $T_N(f_1)T_N(f_2)$ we have*

$$\Lambda_{\alpha_1,\alpha_2,N} = N^{2\alpha_1+2\alpha_2}C_{\alpha_1}C_{\alpha_2}c_1(1)c_2(1)\|K_{\alpha_1,\alpha_2}\| + o(N^{2\alpha_1+2\alpha_2}).$$

with

$$\forall \alpha \in]0, \frac{1}{2}[\ C_\alpha = \frac{\Gamma(1-2\alpha)\sin(\pi\alpha)}{\pi}$$

and K_{α_1,α_2} the integral operator on $L^2[0,1]$ with kernel $(x,y) \to \int_0^1 |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt$.

Then we obtain bounds on $\|K_{\alpha_1,\alpha_2}\|$ which provides bounds on $\Lambda_{\alpha_1,\alpha_2,N}$ (see the theorem 4).

In a second part we apply this result to obtain the maximal eigenvalue $\Lambda_{\alpha,\beta,N}$ of the more general symbols

$$\tilde{f}_1(e^{i\theta}) = |1-e^{i\theta}|^{-2\alpha}\prod_{j=1}^{p}|e^{i\theta_j}-e^{i\theta}|^{-2\alpha_j}c_1(e^{i\theta}) \quad \text{and} \quad \tilde{f}_2(e^{i\theta}) = |1-e^{i\theta}|^{-2\beta}\prod_{j=1}^{q}|e^{i\theta_j}-e^{i\theta}|^{-2\alpha_j}c_2(e^{i\theta}) \tag{1}$$

with $0 < \alpha, \beta < \frac{1}{2}$, $\alpha > \max_{1\leq j\leq p}(\alpha_j)$, $\beta > \max_{1\leq j\leq q}(\beta_j)$ and where c_1, c_2 are two regular functions satisfying precise hypotheses. We obtain

$$\Lambda_{\alpha,\beta,N} \sim CN^{2\alpha+2\beta}\|K_{\alpha,\beta}\|$$

(see Theorem 5 for the expression of C).

Remark 1 *To get Theorem 5 we give in Lemma 2 an asymptotic of the Fourier coefficients of the symbols $\tilde{f}_1$ and $\tilde{f}_2$ of (1). We may observe that this lemma provides a statement that slightly differs from Theorem 4.2. in [8].*

This statement will be

Theorem 2 *Put $\sigma = \prod_{j=1}^{R}|\chi-\chi_j|^{-2\alpha_j}c$ where $\forall j,\ \chi_j \in \mathbb{T}$ and*

i) $0 < \alpha_1 < \frac{1}{2}$

ii) $\alpha_1 > \max_{2\leq j\leq R}(\alpha_j)$.

If c is a regular positive function with $c \in A(\mathbb{T}, r)$ for $1 > r > 0$ (see the point 2.2) we have

$$\Lambda_N \sim H \|K_{\alpha_1}\| N^{2\alpha_1}$$

where Λ_N is the maximal eigenvalue of $T_N(\sigma)$, $H = C_{\alpha_1} c(\chi_1) \prod_{j=2}^{R} |1 - \chi_j| -2\alpha_j$ and K_{α_1} is the integral operator on $L^2(0,1)$ with kernel $(x,y) \to |x-y|^{-2\alpha_1 - 1}$.

An important application of the knowledge of the maximal eigenvalue of the product of two Toeplitz matrices $T_N(f_1)$ and $T_N(f_2)$ is the application of the Gärtner-Ellis Theorem to obtain a large deviation principle and([11]). Here we consider the case of long memory (see also [15]). For the application of the Gärtner-Ellis Theorem in the case where f_1 and f_2 belong to $L^\infty(\mathbb{T})$ [2], [3],[4] are good references.

Remark 2 *For the case where $f, g \in L^\infty(\mathbb{T})$ is it not true in general that the maximal eigenvalue of $T_N(f)T_N(g)$ goes to $\mathrm{essup}(fg)$. Likewise it is not always true that the minimal eigenvalue of $T_N(f)T_N(g)$ goes to $\mathrm{essinf}(fg)$. If we denote these maximal and minimal eigenvalues by $\Lambda_{\max,N}$ and $\Lambda_{\min,N}$ Bercu, Bony and Bruneau give in [4] an example of two functions $f, g \in C^0(\mathbb{T}), g \geq 0$ such that $\lim\limits_{N \to +\infty} \Lambda_{\max,N}$ exists but is greater than $\sup_{\theta \in \mathbb{T}} (fg)(\theta)$ and another example where $\lim\limits_{N \to +\infty} \Lambda_{\min,N}$ is defined but is smaller than $\inf_{\theta \in \mathbb{T}} (fg)(\theta)$. However if $f, g \in L^\infty(\mathbb{T})$ since $\mathrm{essup}(f)\mathrm{essup}(g) - T_N(f)T_N(g)$ is a nonnegative operator it is quite easy to obtain, from the results of [2], that*

$$\mathrm{essup}(f)\mathrm{essup}(g) = \mathrm{essup}(fg) \Rightarrow \lim_{N \to +\infty} \Lambda_{\max,N} = \mathrm{essup}(fg).$$

2 Main result

In the rest of this paper we denote by χ the function $\theta \to e^{i\theta}$.

2.1 Single Fisher-Hartwig singularities.

Theorem 3 *Let $f_1 = |1-\chi|^{-2\alpha_1} c_1$ and $f_2 = |1-\chi|^{-2\alpha_2} c_2$ with $0 < \alpha_1, \alpha_2 < \frac{1}{2}$ and $c_1, c_2 \in L^\infty(\mathbb{T})$ that are continuous and nonzero in 1. We have*

$$\|T_N(f_1)T_N(f_2)\| = N^{2\alpha_1 + 2\alpha_2} C_{\alpha_1} C_{\alpha_2} c_1(1) c_2(1) \|K_{\alpha_1,\alpha_2}\| + o(N^{2\alpha_1 + 2\alpha_2}).$$

with C_{α_1}, C_{α_2} and K_{α_1,α_2} as in Theorem 1.

Now we give a lemma which is useful to prove Theorem 4.

Lemma 1 *There exits a constant $H_{\alpha_1\alpha_2}$ such that for all $(x,y) \in [0,1]^2$, $x \neq y$*

$$|y-x|^{2\alpha_1 + 2\alpha_2 - 1} \leq \int_0^1 |x-t|^{2\alpha_1 - 1} |y-t|^{2\alpha_2 - 1} dt \leq H_{\alpha_1\alpha_2} |x-y|^{2\alpha_1 + 2\alpha_2 - 1},$$

with

$$H_{\alpha_1\alpha_2} = \mathbf{B}(2\alpha_1, 2\alpha_2) + \int_0^{+\infty} (v^{2\alpha_1 - 1}(1+v)^{2\alpha_2 - 1} + v^{2\alpha_2 - 1}(1+v)^{2\alpha_1 - 1}) dv$$

that is also

$$H_{\alpha_1\alpha_2} = \mathbf{B}(2\alpha_1, 2\alpha_2) + \mathbf{B}(2\alpha_2, 3 - 2\alpha_1 - 2\alpha_2) + \mathbf{B}(2\alpha_1, 3 - 2\alpha_1 - 2\alpha_2).$$

Then we have, as corollary of Theorem 3

Theorem 4 *With the hypotheses of Theorem 3, if* $\gamma_{\alpha_1,\alpha_2}$ *is such that* $\|T_N(f_1)T_N(f_2)\| \sim N^{2\alpha_1+2\alpha_2}c_1(1)c_2(1)\gamma_{\alpha_1\alpha_2}$ *we have the bounds*

$$\psi(\alpha_1+\alpha_2)\frac{C_{\alpha_1}C_{\alpha_2}}{C_{\alpha_1+\alpha_2}} \leq \gamma_{\alpha_1,\alpha_2} \leq H_{\alpha_1\alpha_2}\frac{C_{\alpha_1}C_{\alpha_2}}{C_{\alpha_1+\alpha_2}}\frac{1}{\alpha_1+\alpha_2},$$

with $\psi(\alpha) = \frac{1}{2\alpha}\left(\frac{2}{4\alpha+1} + 2\frac{\Gamma^2(2\alpha+1)}{\Gamma(4\alpha+2)}\right)^{\frac{1}{2}}$.

If we consider now the two symbols $f_{1,\chi_0} = |\chi_0-\chi|^{-2\alpha_1}c_1$ and $f_{2,\chi_0} = |\chi_0-\chi|^{-2\alpha_2}c_2$ with $\chi_0 \in \mathbb{T}$ it is known (see [14]) that

$$T_N(|\chi_0-\chi|^{-2\alpha}c) = \Delta_0(\chi_0)T_N\left(|1-\chi|^{-2\alpha}c_{\chi_0}\right)\Delta_0^{-1}(\chi_0)$$

where $c_{\chi_0}(\chi) = c(\chi_0\chi)$ and where $\Delta_0(\chi_0)$ is the diagonal matrix defined by $(\Delta_0(\chi_0))_{i,j} = 0$ if $i \neq j$ and $(\Delta_0(\chi_0))_{i,i} = \chi_0^i$. Hence we have the following corollary of Theorems 3 and 4

Corollary 1 *With the previous notations and hypotheses we have*

$$\|T_N(f_{1,\chi_0})T_N(f_{2,\chi_0})\| \sim N^{-2\alpha_1-2\alpha_2}C_{\alpha_1}C_{\alpha_2}c_1(\chi_0)c_2(\chi_0)\|K_{\alpha_1,\alpha_2}\|$$

2.2 Several Fisher-Hartwig singularities

Let $r > 0$, we denote by $A(\mathbb{T},r)$ the set $\{g \in L^1(\mathbb{T}) \mid \sum_{u\in\mathbb{Z}}|u|^r|\hat{g}(u)| < \infty\}$. We first state the following lemma

Lemma 2 *Put* $\sigma = \prod_{j=1}^{R}|\chi-\chi_j|^{-2\alpha_j}c$ *where* $\forall j,\ \chi_j \in \mathbb{T}$ *and* $\alpha_1 > \max_{2\leq j\leq R}(\alpha_j)$. *If* c *is a regular positive function with* $c \in A(\mathbb{T},r)$ *(*$1 \geq r > 0$ *if* $\frac{1}{2} > \alpha_1 > 0$ *and* $r \geq 2$ *if* $0 > \alpha_1 > -\frac{1}{2}$*) we have*

$$\widehat{\sigma}(M) = C_{\alpha_1}c(\chi_1)\prod_{j=2}^{R}|1-\chi_j|^{-2\alpha_j}M^{2\alpha_1-1} + o(M^{2\alpha_1-1})$$

uniformly in M.

This lemma and the proof of Theorem 3 allow us to obtain

Theorem 5 *Let* $\tilde{f}_1 = |1-\chi|^{-2\alpha}\prod_{j=1}^{p}|\chi_j-\chi|^{-2\alpha_j}c_1$ *and* $\tilde{f}_2 = |1-\chi|^{-2\beta}\prod_{j=1}^{q}|\tilde{\chi}_j-\chi|^{-2\alpha_j}c_2$ *with* $0 < \alpha,\beta < \frac{1}{2}$, $\alpha > \max_{1\leq j\leq p}(\alpha_j)$, $\beta > \max_{1\leq j\leq q}(\beta_j)$, $\chi_j \neq 1$, $\tilde{\chi}_j \neq 1$ *and* c_1, c_2 *two regular functions with* $c_1 \in A(\mathbb{T},r_1), c_2 \in A(\mathbb{T},r_2)$ *for* $1 \geq r_1, r_2 > 0$. *Then*

$$\|T_N(\tilde{f}_1)T_N(\tilde{f}_2)\| \sim CN^{2\alpha+2\beta}\|K_{\alpha,\beta}\|$$

with

$$C = c_1(1)c_2(1)C_\alpha C_\beta\prod_{j=1}^{p}|1-\chi_j|^{+2\alpha_j}\prod_{j=1}^{q}|1-\tilde{\chi}_j|^{+2\beta_j}.$$

With the same hypotheses on α and β we can now consider $T_N(\tilde{f}_{1,\chi_0})T_N(\tilde{f}_{2,\chi_0})$ with $\tilde{f}_{1,\chi_0} = |\chi_0 - \chi|^{-2\alpha} \prod_{j=1}^{p} |\chi_j - \chi|^{-2\alpha_j} c_1$ and $\tilde{f}_{2,\chi_0} = |\chi_0 - \chi|^{-2\beta} \prod_{j=1}^{q} |\tilde{\chi}_j - \chi|^{-2\alpha_j} c_2$, with $\forall j \in \{1, \cdots, p\}$ $\chi_j \neq \chi_0$ and $\forall h \in \{1, \cdots, q\}$ $\tilde{\chi}_h \neq \chi_0$. We obtain the corollary

Corollary 2 *With the previous notations and hypotheses we have*

$$\|T_N(\tilde{f}_{1,\chi_0})T_N(\tilde{f}_{2,\chi_0})\| \sim N^{2\alpha+2\beta} C_{\chi_0} \|K_{\alpha_1,\alpha_2}\|$$

with

$$C_{\chi_0} = C_\alpha C_\beta c_1(\chi_0) c_2(\chi_0) \prod_{j=1}^{p} |\chi_0 - \chi_j|^{-2\alpha_j} \prod_{j=1}^{q} |\chi_0 - \chi_j|^{-2\beta_j}.$$

3 Demonstration of Theorem 3

Let us recall the following Widom's result (see, for instance, [9]).

Lemma 3 *Let $A_N = (a_{i,j})_{i,j=0}^{N-1}$ be an $N \times N$ matrix with complex entries. We denote by G_N the integral operator on $L^2[0,1]$ with kernel*

$$g_N(x,y) = a_{[Nx],[Ny]}, \quad (x,y) \in (0,1)^2.$$

Then the spectral norm of A_N and the operator norm of G_N are related by the equality $\|A_N\| = N\|G_N\|$.

Denote by K_N and K_{α_1,α_2} the integral operators on $L^2(0,1)$ with the kernels, defined for $x \neq y$ by

$$k_N(x,y) = N^{-2\alpha_1-2\alpha_2+1} \sum_{0 \leq u \leq N, u \neq [Nx], u \neq [Ny]} \Big| [Nx] - u \Big|^{2\alpha_1 - 1} \Big| [Ny] - u \Big|^{2\alpha_2 - 1}$$

and

$$k_{\alpha_1,\alpha_2}(x,y) = \int_0^1 |x-t|^{2\alpha_1-1} |y-t|^{2\alpha_2-1} dt.$$

To prove Theorem 3 we first assume that the following lemma is true.

Lemma 4 *The operator K_N converges to K_{α_1,α_2} in the operator norm on $L^2(0,1)$.*

Assume Lemma 4 is true. Suppose $c_1 = c_2 = 1$. Then put $T_{1,N}, T_{2,N}, D_{1,N}, D_{2,N}$ the $(N+1) \times (N+1)$ matrices defined by if $k \neq l$

$$(T_{1,N})_{(k+1,l+1)} = C_{\alpha_1} |k-l|^{2\alpha_1 - 1} \quad (T_{2,N})_{(k+1,l+1)} = C_{\alpha_2} |k-l|^{2\alpha_2 - 1},$$

and $(T_{1,N})_{(k+1,k+1)} = 0, (T_{2,N})_{(k+1,k+1)} = 0$. On the other hand

$$(D_{1,N})_{(k+1,l+1)} = (T_N(f_1))_{(k+1,l+1)} - (T_{1,N})_{(k+1,l+1)},$$

$$(D_{2,N})_{(k+1,l+1)} = (T_N(f_2))_{(k+1,l+1)} - (T_{2,N})_{(k+1,l+1)}.$$

We can remark that $D_{1,N}$ and $D_{2,N}$ are Toeplitz matrices such that $(D_{1,N})_{(k+1,l+1)} = o(|k-l|^{2\alpha_1-1})$ and $(D_{2,N})_{(k+1,l+1)} = o(|k-l|^{2\alpha_2-1})$ (see[10]) and this implies (see [8])

$$\|D_{1,N}\| = o(N^{2\alpha_1}) \quad \text{and} \quad \|D_{2,N}\| = o(N^{2\alpha_2}).$$

Then we have the upper bound

$$\|T_N(f_1)T_N(f_2) - T_{1,N}T_{2,N}\| \leq \|D_{1,N}T_{2,N}\| + \|D_{2,N}T_{1,N}\| + \|D_{1,N}D_{2,N}\|$$

and (see [8])

$$\|D_{1,N}T_{2,N}\| \leq \|D_{1,N}\|\|T_{2,N}\| = o(N^{2\alpha_1})O(N^{2\alpha_2}) = o(N^{2\alpha_1+2\alpha_2})$$

$$\|D_{2,N}T_{1,N}\| \leq \|D_{2,N}\|\|T_{1,N}\| = o(N^{2\alpha_2})O(N^{2\alpha_1}) = o(N^{2\alpha_1+2\alpha_2}).$$

$$\|D_{1,N}D_{2,N}\| \leq \|D_{1,N}\|\|D_{2,N}\| = o(N^{2\alpha_1})o(N^{2\alpha_2}) = o(N^{2\alpha_1+2\alpha_2}).$$

Hence

$$\|T_N(f_1)T_N(f_2)\| = \|T_{1,N}T_{2,N}\| + o(N^{2\alpha_1+2\alpha_2}).$$

Lemma 3 implies

$$\left\|\frac{T_{1,N}T_{2,N}}{N}\right\| = \|N^{2\alpha_1+2\alpha_2-1}K_N\|$$

and with Lemma 4 we obtain $\lim_{N\to+\infty} \|K_N\| = \|K_{\alpha_1\alpha_2}\|$ that ends the proof in the case where the regular function equals 1. Now assume that c_1, c_2 are any continuous positive functions in $L^\infty(\mathbb{T})$. Let $\tilde{c}_1$ and $\tilde{c}_2$ defined by $\forall j \in \{1,2\} \quad \tilde{c}_j(\theta) = c_j(\theta)$ if $\theta \neq 1$ and $\tilde{c}_j(1) = 0$. If $\tilde{f}_1 = |1-\chi|^{-2\alpha_1}\tilde{c}_1$ and $\tilde{f}_2 = |1-\chi|^{-2\alpha_2}\tilde{c}_2$ we have (see [8])

$$\|T_N\tilde{f}_1\| = o(N^{2\alpha_1}) \quad \|\tilde{T}_N\tilde{f}_2\| = o(N^{2\alpha_2}).$$

Hence $\|T_N\tilde{f}_1 T_N\tilde{f}_2\| = o(N^{-2\alpha_1-2\alpha_2})$. Since $f_1 = |1-\chi|^{-2\alpha_1}(\tilde{c}_1 + c_1(1))$ and $f_2 = |1-\chi|^{-2\alpha_2}(\tilde{c}_2 + c_2(1))$ we have

$$\|T_N(f_1)T_N(f_2) - T_N\left(c_1(1)|1-\chi|^{-2\alpha_1}\right)T_N\left(c_2(1)|1-\chi|^{-2\alpha_2}\right)\| = o(N^{2\alpha_1+2\alpha_2})$$

and we finally get, via the beginning of the proof

$$\|T_N(f_1)T_N(f_2)\| = N^{2\alpha_1+2\alpha_2}C_{\alpha_1}C_{\alpha_2}c_1(1)c_2(1)\|K_{\alpha_1\alpha_2}\| + o(N^{2\alpha_1+2\alpha_2})$$

which is the expected formula. We are therefore left with proving Lemma 4.
<u>Proof of the lemma 4</u>:

Fix μ, $0 < \mu < 1$ sufficiently close to 1 such that $\mu > \max(1-2\alpha_1, 1-2\alpha_2, \frac{1}{2})$. Put

$$k_N^1(x,y) = \begin{cases} k_N(x,y) & \text{if} \quad |x-y| > N^{\mu-1}, \\ 0 & \text{otherwise} \end{cases}$$

$$k_N^2(x,y) = \begin{cases} k_N(x,y) & \text{if} \quad |x-y| < N^{\mu-1}, \\ 0 & \text{otherwise} \end{cases}$$

$$k_{\alpha_1,\alpha_2,N}^1(x,y) = \begin{cases} k_{\alpha_1,\alpha_2}(x,y) & \text{if} \quad |x-y| > N^{\mu-1}, \\ 0 & \text{otherwise.} \end{cases}$$

$$k^2_{\alpha_1,\alpha_2,N}(x,y) = \begin{cases} k_{\alpha_1,\alpha_2}(x,y) & \text{if} \quad |x-y| < N^{\mu-1}, \\ 0 & \text{otherwise.} \end{cases}$$

If we denote by $K^1_N, K^1_{\alpha_1,\alpha_2,N}, K^2_{\alpha_1,\alpha_2,N}$ the integral operator on $L^2(0,1)$ with the kernels h^1_N, h^1_N, $k^1_N k^1_{\alpha_1,\alpha_2,N}, k^2_{\alpha_1,\alpha_2,N}$ respectively. We have

$$\|K_{\alpha_1,\alpha_2} - K_N\| \leq \|K^1_{\alpha_1,\alpha_2,N} - K^1_N\| + \|K^2_{\alpha_1,\alpha_2,N}\| + \|K^2_N\|.$$

Hence we have to show that

$$\lim_{N\to+\infty} \|K^1_{\alpha_1,\alpha_2,N} - K^1_N\| = 0, \lim_{N\to+\infty} \|K^2_{\alpha_1,\alpha_2,N}\| = 0, \lim_{N\to+\infty} \|K^2_N\| = 0.$$

First we prove the following lemma.

Lemma 5 *When N goes to the infinity $\|K^1_{\alpha_1,\alpha_2,N} - K^1_N\| \to 0$*

Proof : To prove that $\|K^1_{\alpha_1,\alpha_2,N} - K^1_N\| \to 0$ it suffices to show that $|k^1_N(x,y) - k^1_{\alpha_1,\alpha_2}|$ converges uniformly to zero for $|x-y| > N^{\mu-1}$. First we may assume that $x < y$ and we consider the case $[Nx] > N^\mu$ and $[Ny] < N - N^\mu$. Next we study the cases $[Nx] \leq N^\mu$ and $[Ny] \geq N - N^\mu$.

For $|x-y| > N^{\mu-1}$ we have to consider the difference

$$S_N(x,y) = \frac{1}{N} \sum_{u=0, u\neq[Nx], u\neq[Ny]}^{N} \left|\frac{[Nx]}{N} - \frac{u}{N}\right|^{2\alpha_1-1} \left|\frac{[Ny]}{N} - \frac{u}{N}\right|^{2\alpha_2-1}$$
$$- \int_0^1 |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1} dt.$$

Let $S_{i,N}(x,y)$, $1 \leq i \leq 7$ be the following differences

$$S_{1,N}(x,y) = \frac{1}{N} \sum_{u=0}^{[Nx]-N^{\mu_1}} \left|\frac{[Nx]}{N} - \frac{u}{N}\right|^{2\alpha_1-1} \left|\frac{[Ny]}{N} - \frac{u}{N}\right|^{2\alpha_2-1}$$
$$- \int_0^{\frac{[Nx]}{N} - N^{\mu_1-1}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1} dt,$$

$$S_{2,N}(x,y) = \frac{1}{N} \sum_{[Nx]-N^{\mu_1}+1}^{[Nx]-1} \left|\frac{[Nx]}{N} - \frac{u}{N}\right|^{2\alpha_1-1} \left|\frac{[Ny]}{N} - \frac{u}{N}\right|^{2\alpha_2-1}$$
$$- \int_{\frac{[Nx]}{N} - N^{\mu_1-1} + \frac{1}{N}}^{\frac{[Nx]-1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1} dt,$$

$$S_{3,N}(x,y) = \frac{1}{N} \sum_{[Nx]+1}^{[Nx]+N^{\mu_2}} \left|\frac{[Nx]}{N} - \frac{u}{N}\right|^{2\alpha_1-1} \left|\frac{[Ny]}{N} - \frac{u}{N}\right|^{2\alpha_2-1}$$
$$- \int_{\frac{[Nx]+1}{N}}^{\frac{[Nx]}{N} + N^{\mu_2-1}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1} dt,$$

$$S_{4,N}(x,y) = \frac{1}{N}\sum_{[Nx]+1+N^{\mu_2}}^{[Ny]-N^{\mu_3}} \Big|\frac{[Nx]}{N} - \frac{u}{N}\Big|^{2\alpha_1-1}\Big|\frac{[Ny]}{N} - \frac{u}{N}\Big|^{2\alpha_2-1}$$
$$- \int_{\frac{[Nx]+1}{N}+N^{\mu_2-1}}^{\frac{[Ny]}{N}-N^{\mu_3-1}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt,$$

$$S_{5,N}(x,y) = \frac{1}{N}\sum_{[Ny]-N^{\mu_3}+1}^{[Ny]-1} \Big|\frac{[Nx]}{N} - \frac{u}{N}\Big|^{2\alpha_1-1}\Big|\frac{[Ny]}{N} - \frac{u}{N}\Big|^{2\alpha_2-1}$$
$$- \int_{\frac{[Ny]+1}{N}-N^{\mu_3-1}}^{\frac{[Ny]-1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt,$$

$$S_{6,N}(x,y) = \frac{1}{N}\sum_{[Ny]+1}^{[Ny]+N^{\mu_4}} \Big|\frac{[Nx]}{N} - \frac{u}{N}\Big|^{2\alpha_1-1}\Big|\Big|\frac{[Ny]}{N} - \frac{u}{N}\Big|^{2\alpha_2-1}$$
$$- \int_{\frac{[Ny]+1}{N}}^{\frac{[Ny]}{N}+N^{\mu_4-1}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt,$$

$$S_{7,N}(x,y) = \frac{1}{N}\sum_{[Ny]+N_4^{\mu}+1}^{N} \Big|\frac{[Nx]}{N} - \frac{u}{N}\Big|^{2\alpha_1-1}\Big|\frac{[Ny]}{N} - \frac{u}{N}\Big|^{2\alpha_2-1}$$
$$- \int_{\frac{[Ny]+1}{N}+N^{\mu_4-1}}^{1} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt$$

with $0<\mu_1<\mu$, $0<\mu_2<\mu$, $0<\mu_3<\mu$, $0<\mu_4<\mu$. We can remark that

$$S_{1,N}(x,y) \sim \int_0^{\frac{[Nx]}{N}-N^{\mu_1-1}} \left(\frac{[Nx]}{N}-t\right)^{2\alpha_1-1}\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} - (x-t)^{2\alpha_1-1}(y-t)^{2\alpha_2-1}\,dt.$$

We may study the two differences

$$S'_{1,N}(x,y) = \int_0^{\frac{[Nx]}{N}-N^{\mu_1-1}} \left(\left(\frac{[Nx]}{N}-t\right)^{2\alpha_1-1} - (x-t)^{2\alpha_1-1}\right)\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} dt$$

and

$$S''_{1,N}(x,y) = \int_0^{\frac{[Nx]}{N}-N^{\mu_1-1}} (x-t)^{2\alpha_1-1}\left(\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} - (y-t)^{2\alpha_2-1}\right)dt.$$

Since $|\frac{[Nx]-x}{x-t}| \le N^{-\mu_1}$ we have $\left(\frac{[Nx]}{N}-t\right)^{2\alpha_1-1} - (x-t)^{2\alpha_1-1} = O(N^{-\mu_1})$ and

$$|S'_{1,N}(x,y)| \le O(N^{-\mu_1})\int_0^{\frac{[Nx]}{N}-N^{\mu_1-1}} \left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} dt = O(N^{-\mu_1}) = o(1).$$

The same method provides

$$S''_{1,N}(x,y)| = O(N^{-\mu}) = o(1).$$

As previously we have now

$$S_{2,N}(x,y) \sim \int_{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}}^{\frac{[Nx]-1}{N}} \left(\left(\frac{[Nx]}{N} - t \right)^{2\alpha_1-1} \left(\frac{[Ny]}{N} - t \right)^{2\alpha_2-1} \right. \\ \left. - (x-t)^{2\alpha_1-1} (y-t)^{2\alpha_2-1} \right) dt.$$

Obviously we have to consider the differences

$$S'_{2,N}(x,y) = \int_{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}}^{\frac{[Nx]-1}{N}} \left(\left(\frac{[Nx]}{N} - t \right)^{2\alpha_1-1} - (x-t)^{2\alpha_1-1} \right) \left(\frac{[Ny]}{N} - t \right)^{2\alpha_2-1} dt$$

and

$$S''_{2,N}(x,y) = \int_{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}}^{\frac{[Nx]-1}{N}} (x-t)^{2\alpha_1-1} \left(\left(\frac{[Ny]}{N} - t \right)^{2\alpha_2-1} - (y-t)^{2\alpha_2-1} \right) dt.$$

With the main value theorem we can write

$$S'_{2,N}(x,y) = -(-2\alpha_1+1) \left(\frac{[Nx]}{N} - x \right) \int_{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}}^{\frac{[Nx]-1}{N}} c_{x,N}^{2\alpha_1-2}(t) \left(\frac{[Ny]}{N} - t \right)^{2\alpha_2-1} dt$$

with $c_{x,N}(t) > N^{-1}$ and

$$\int_{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}}^{\frac{[Nx]-1}{N}} \left(\frac{[Ny]}{N} - t \right)^{2\alpha_2-1} dt = O(N^{\mu_1-\mu}).$$

So $S'_{2,N}(x,y) = O(N^{\mu_1-\mu)-2\alpha_1+1})$. We can remark that $-2\alpha_1+1-\mu<0 \iff -2\alpha_1+1<\mu$. Hence if $-2\alpha_1+1<\mu$ and μ_1 sufficiently little we have $S'_{2,N}(x,y) = o(1)$. Likewise we have $S''_{2,N}(x,y) = O\left(N^{-1+(\mu-1)(2\alpha_2-2)}\right)$. Hence $\mu > \frac{-2\alpha_2+1}{-2\alpha_2+2} \Rightarrow S''_{2,N}(x,y) = o(1)$, and since $-2\alpha_2+1 > \frac{-2\alpha_2+1}{-2\alpha_2+2}$ we have $S''_{2,N}(x,y) = o(1)$.

We prove exactly as previously

$$\mu > -2\alpha_1+1 \quad \text{and} \quad \mu > -2\alpha_2+1 \Rightarrow S_{3,N} = o(1)$$
$$\mu_2 > 0 \quad \text{and} \quad \mu_3 > 0 \Rightarrow S_{4,N} = o(1)$$

Swapping x and y we obtain

- $\mu > -2\alpha_2+1$ and $\mu > \frac{-2\alpha_1+1}{-2\alpha_1+2}$ then $S_{5,N}(x,y) = o(1)$.
- $\mu > -2\alpha_2+1$ and $\mu > \frac{-2\alpha_1+1}{-2\alpha_1+2}$ then $S_{6,N}(x,y) = o(1)$.
- $\mu > 0$ and $\mu_4 > 0$ then $S_{7,N}(x,y) = o(1)$.

To complete the proof we have still to bound the integrals

$$\int_{\frac{[Nx]}{N}-N^{\mu_1-1}}^{\frac{[Nx]}{N}-N^{\mu_1-1}+\frac{1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt, \quad \int_{\frac{[Nx]-1}{N}}^{\frac{[Nx]+1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt$$

$$\int_{\frac{[Nx]}{N}+N^{\mu_2-1}}^{\frac{[Nx]}{N}+N^{\mu_2-1}+\frac{1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt, \quad \int_{\frac{[Ny]}{N}-N^{\mu_3-1}}^{\frac{[Ny]}{N}-N^{\mu_3-1}+\frac{1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt$$

$$\int_{\frac{[Ny]-1}{N}}^{\frac{[Ny]+1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt, \quad \int_{\frac{[Ny]}{N}+N^{\mu_4-1}}^{\frac{[Ny]}{N}+N^{\mu_4-1}+\frac{1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt$$

which are obviously in $o(1)$with the hypotheses on μ.

Assume now $1-N^{\mu-1} > y > N^{\mu-1} > x > 0$. For this case we have to consider the decomposition $S_N(x,y) = \sum_{i=1}^{6} S_{i,N}(x,y)$ with

$$S_{1,N}(x,y) = \frac{1}{N}\sum_{u=0}^{[Nx]-1} \left|\frac{[Nx]}{N}-\frac{u}{N}\right|^{2\alpha_1-1}\left|\frac{[Ny]}{N}-\frac{u}{N}\right|^{2\alpha_2-1} - \int_0^{\frac{[Nx]-1}{N}} |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt,$$

and $S_{i,N}$ defined as $S_{i+1,N}$ in the previous case. We still consider the two differences

$$S'_{1,N}(x,y) = \int_0^{\frac{[Nx]-1}{N}} \left(\left(\frac{[Nx]}{N}-t\right)^{2\alpha_1-1} - (x-t)^{2\alpha_1-1}\right)\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} dt$$

and

$$S''_{1,N}(x,y) = \int_0^{\frac{[Nx]-1}{N}} (x-t)^{2\alpha_1-1}\left(\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} - (y-t)^{2\alpha_2-1}\right) dt.$$

We have

$$S'_{1,N}(x,y) \leq N^{(\mu-1)(2\alpha_2-1)}O\left((\frac{[Nx]}{N})^{2\alpha_1} - x^{2\alpha_1}\right) = N^{(\mu-1)(2\alpha_2-1)}O(N^{-2\alpha_1}).$$

We can remark that $(\mu-1)(2\alpha_2-1)-2\alpha_1 < 0 \iff \mu > \frac{2\alpha_1}{2\alpha_2-1}+1$. Since $1-2\alpha_1 > \frac{2\alpha_1}{2\alpha_2-1}+1$ the hypotheses on μ give $S'_{1,N}(x,y) = o(1)$. Moreover $\left(\left(\frac{[Ny]}{N}-t\right)^{2\alpha_2-1} - (y-t)^{2\alpha_2-1}\right) = O(N^{-\mu})$ and $S''_{1,N}(x,y) = O(N^{-\mu}) = o(1)$. The differences $S_{i,N}$ for $2 \leq i \leq 6$ are as in the first case.

The case $N - N^{\mu} < [Ny] < N$ can be tackled identically.

□

Proof of $\|K^2_{\alpha_1,\alpha_2,N}\| \to 0$

From the lemma 1 we have, for $g \in L^2(\mathbb{T})$ et $y \in [0,1]$

$$\begin{aligned} K^2_{\alpha_1,\alpha_2,N}(g)(x) = & \int_0^1 k_{\alpha_1,\alpha_2}(x,y)g(y)dy = \int_{x-N^{\mu-1}}^{x+N^{\mu+1}} f_{\alpha_1,\alpha_2}(x,y)g(y)dy \\ \leq & \int_{x-N^{\mu-1}}^{x+N^{\mu+1}} H_{\alpha_1,\alpha_2}|x-y|^{2\alpha_1+2\alpha_2-1}g(y)dy = K'^2_{\alpha_1,\alpha_2,N}(g)(y) \end{aligned}$$

where $K'^2_{\alpha_1,\alpha_2,N}$ is the integral operator on $L^2(0,1)$ with kernel

$$k'_{\alpha_1,\alpha_2,N}(x,y) = H_{\alpha_1,\alpha_2}|x-y|^{2\alpha_1 2\alpha_2-1}$$

if $|x-y| < N^{\mu-1}$ and $k_N^{\alpha_1,\alpha_2,\prime}(x,y) = 0$ otherwise.
If $\|g\|_2 = 1$we have

$$\int_0^1 |K^2_{\alpha_1,\alpha_2,N}(g)(x)|^2 dx = = \int_0^1 \Big| \int_0^1 k_{\alpha_1,\alpha_2,N}(x,y) g(y) dy \Big|^2 dx$$
$$\leq \int_0^1 \Big(\int_0^1 k_{\alpha_1,\alpha_2,N}(x,y) |g(y)| dy \Big)^2 dx$$
$$\leq \int_0^1 \Big(\int_0^1 k'_{\alpha_1,\alpha_2,N}(x,y) |g(y)| dy \Big)^2 dx \leq \|K'^2_{\alpha_1,\alpha_2,N}\|^2.$$

Hence $\|K^2_{\alpha_1,\alpha_2,N}\| \leq \|K'^2_{\alpha_1,\alpha_2,N}\| = O\left(N^{(\mu-1)(2\alpha_1 2\alpha_2)}\right) = o(1)$ (see [8]).

Proof of $\|K_N^2\| \to 0$

As in [8] we define the integral operator $\tilde{K}_N^2$ on $L^2(0,1)$ with the kernel $\tilde{k}_N^2$ defined by k_N^2 in the staircase-like bordered strip $|[Nx]-[Ny]| < N^\mu$ and be zero otherwise. On the squares where $\tilde{k}^2_N(x,y) - k_N^2(x,y) \neq 0$ we have $|[Nx]-[Ny]| \sim N^\mu$ and, as for the proof of the lemma 4, for (x,y) in this squares

$$\tilde{k}^2_N(x,y) - k^2_N(x,y) \sim \int_0^1 |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1} dt$$

and always with the lemma 1

$$|\tilde{k}^2_N(x,y) - k^2_N(x,y)| \leq H_{\alpha_1,\alpha_2}|x-y|^{2\alpha_1+2\alpha_2-1} = O(N^{(\mu-1)(2\alpha_1+2\alpha_2-1)}).$$

As the difference $\tilde{h}^2_N(x,y) - h^2_N(x,y)$ is supported in about $4(N-N^\mu) = O(N)$ squares of side length $\frac{1}{N}$ we have the squared Hilbert-Schmidt norm

$$\|\tilde{K}^2_N - K^2_N\| = O\left(N \frac{1}{N^2} N^{(\mu-1)(4\alpha_1+4\alpha_2-2)}\right).$$

If $2\alpha_1+2\alpha_2-1>0$ we have $(\mu-1)(4\alpha_1+4\alpha_2-2)-1<0$ and

$$\|\tilde{K}^2_N - K^2_N\| \to 0. \tag{2}$$

Otherwise since $\mu > \frac{1}{2} > \frac{-4\alpha_1-4\alpha_2+1}{-4\alpha_1-4\alpha_2+2}$ we have also (2).

We are therefore with proving $\|\tilde{K}^2_N\| \to 0$. Let B_N be the matrix such $(B_N)_{k+1,l+1} = C_{\alpha_1}C_{\alpha_2} \sum_{u=0,u\neq k,u\neq l} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1}$ if $|k-l| \leq N^\mu$ and $(B_N)_{k+1,l+1} = 0$ otherwise. We have to prove the following technical lemma

Lemma 6 *$\exists M_{\alpha_1,\alpha_2} > 0$ such for $k \neq l$*

$$B_{k+1,l+1} \leq M_{\alpha_1,\alpha_2}|k-l|^{2\alpha_1+2\alpha_2-1}$$

Proof :

Assume $l > k$ and write

$$\sum_{u=0,u\neq k,u\neq l}^{N} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} = \sum_{u=0,}^{k-1} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} +$$
$$+ \sum_{k+1}^{l-1} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} + \sum_{l+1}^{N} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1}.$$

The Euler and Mac-Laurin formula provides

$$\begin{aligned}
&\sum_{u=0,u\neq k}^{k-1} |k-u|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} = \\
&= \int_0^{k-1} (k-u)^{2\alpha_1-1}(l-u)^{2\alpha-1}du + \frac{1}{2}\left((l-k+1)^{2\alpha_2-1} + k^{2\alpha_1-1}l^{2\alpha_2-1}\right)(1+o(1)).
\end{aligned}$$

Since $2\alpha_1 - 1 < 0$ and $2\alpha_2 - 1 < 0$ one can find easily $M_1 > 0$ such that

$$\left((l-k+1)^{2\alpha_2-1} + k^{2\alpha_1-1}l^{2\alpha_2-1}\right) < M_1(l-k)^{2\alpha_1+2\alpha_2-1}.$$

And we have also

$$\begin{aligned}
\int_0^{k-1} (k-u)^{2\alpha_1-1}(l-u)^{2\alpha-2}du =&(l-k)^{2\alpha_1+2\alpha_2-1}\int_1^{\frac{k}{l-k}} u^{2\alpha_1-1}(1+u)^{2\alpha_2-1}du \\
\leq&(l-k)^{2\alpha_1+2\alpha_2-1}\int_1^{+\infty} u^{2\alpha_1-1}(1+u)^{2\alpha_2-1}du.
\end{aligned}$$

Analogously one can show that

$$\begin{aligned}
&\sum_{u=k+1}^{l-1} |u-k|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} = \\
&= \int_{k+1}^{l-1} (u-k)^{2\alpha_1-1}(l-u)^{2\alpha-1}du + \frac{1}{2}\left((l-k-1)^{2\alpha_2-2} + (l-k-1)^{2\alpha_1-1}\right)(1+o(1))
\end{aligned}$$

and

$$\begin{aligned}
\int_{k+1}^{l-1} (u-k)^{2\alpha_1-1}(l-u)^{2\alpha-1}du &= (l-k)^{2\alpha_2-1}\int_1^{l-k-1} v^{2\alpha_1-1}\left(1-\frac{v}{l-k}\right)^{2\alpha_2-1}dv \\
&= (l-k)^{2\alpha_1+2\alpha_2-1}\int_{\frac{1}{l-k}}^{1-\frac{1}{l-k}} w^{2\alpha_1-1}(1-w)^{2\alpha_2-1}dw \\
&\leq (l-k)^{2\alpha_1+2\alpha_2-1}\int_0^1 w^{2\alpha_1-1}(1-w)^{2\alpha_2-1}dw.
\end{aligned}$$

The last sum provides

$$\begin{aligned}
\sum_{u=l+1}^{N} |u-k|^{2\alpha_1-1}|l-u|^{2\alpha_2-1} = &\int_{l+1}^{N} (u-k)^{2\alpha_1-1}(u-l)^{2\alpha-1}du \\
&+ \frac{1}{2}\left((l-k+1)^{2\alpha_1-2} + (N-k)^{2\alpha_1-1}(N-k)^{2\alpha_2-1}\right)(1+o(1)).
\end{aligned}$$

We have

$$(N-k)^{2\alpha_1-1}(N-k)^{2\alpha_2-1} \leq (l-k)^{2\alpha_1+2\alpha_2-2} \leq (l-k)^{2\alpha_1+2\alpha_2-1}$$

and

$$\begin{aligned}\int_{l+1}^{N}(u-k)^{2\alpha_1-1}(u-l)^{2\alpha-1}du &= (l-k)^{2\alpha_2-1}\int_{l+1-k}^{N-k}v^{2\alpha_1-1}\left(\frac{v}{l-k}-1\right)^{2\alpha_2-1}dv \\ &= (l-k)^{2\alpha_1+2\alpha_2-1}\int_{1+\frac{1}{l-k}}^{\frac{N-k}{l-k}}w^{2\alpha_1-1}(w-1)^{2\alpha_2-1}dw \\ &\leq (l-k)^{2\alpha_1+2\alpha_2-1}\int_{1}^{+\infty}w^{2\alpha_1-1}(w-1)^{2\alpha_2-1}dw\end{aligned}$$

that ends the proof of the lemma. □

Using lemma 3 we can write

$$\|\tilde{H}_N^2\| = \frac{1}{N}N^{2\alpha_1+2\alpha_2+1}\|B_N\|. \tag{3}$$

Consider now the matrix C_N defined by $(C_N)_{k+1,l+1} = 0$ for $|k-l| \geq N^{\mu}$, $(C_N)_{k+1,l+1} = M_{\alpha_1\alpha_2}|k-l|^{-2\alpha_1-2\alpha_2-1}$ for $0 < |k-l| < N^{\mu}$, $(C_N)_{k+1,k+1} = C_{\alpha_1}C_{\alpha_2}\sum_{u=0}^{\infty}u^{-2\alpha_1-2\alpha_2-2}$ if $-2\alpha_1-2\alpha_2-1<0$, $(C_N)_{k+1,k+1} = 2N^{-2\alpha_1-2\alpha_2-1}\int_0^1|\frac{k}{N}-t|^{-2\alpha_1-2\alpha_2-2}dt$ if $-2\alpha_1-2\alpha_2-1>0$. If $x(x_1,\cdots,x_{N+1})$ and $y(y_1,\cdots,y_{N+1})$ are two vectors of $\mathbb{R}^{N+1}$ we have

$$\begin{aligned}\left|\left\langle B_N(x)|y\right\rangle\right| &= \Big|\sum_{i=1}^{N+1}\left(\sum_{j=1}^{N+1}(B_N)_{i,j}\,x_j\right)y_i\Big| \\ &\leq \sum_{i=1}^{N+1}\left(\sum_{j=1}^{N+1}(B_N)_{i,j}\,|x_j|\right)|y_i| \\ &\leq \sum_{i=1}^{N+1}\left(\sum_{j=1}^{N+1}(C_N)_{i,j}\,|x_j|\right)|y_i|\end{aligned}$$

and

$$\|B_N\| \leq \|C_N\|.$$

But

$$\|C_N\| \leq O(\sum_{i=1}^{N^{\mu}} i^{-2\alpha_1-2\alpha_2-1} = O\left(N^{\mu(-2\alpha_1-2\alpha_2)}\right).$$

and from the equality (3)

$$\|\tilde{K}^2_{\alpha_1,\alpha_2,N}\| = O\left(N^{(2\alpha_1+2\alpha_2)(1-\mu)}\right)$$

hence $\|\tilde{K}^2_{\alpha_1,\alpha_2,N}\| \to 0$ that achieves the proof of the lemma 4. □

4 Demonstration of Lemma 1 and Theorem4

4.1 Proof of Lemma 1

Assume $y > x$. We have

$$\begin{aligned}
\int_0^x (x-t)^{2\alpha_1-1}(y-t)^{2\alpha_2-1}dt &= \int_0^x u^{2\alpha_1-1}(y-x+u)^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_2-1}\int_0^x u^{2\alpha_1-1}(1+\frac{u}{y-x})^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^{\frac{x}{y-x}} v^{2\alpha_1-1}(1+v)^{2\alpha_2-1}dv.
\end{aligned}$$

Consequently

$$\int_0^x (x-t)^{2\alpha_1-1}(y-t)^{2\alpha_2-1}dt \le (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^\infty v^{2\alpha_1-1}(1+v)^{2\alpha_2-1}dv.$$

We can also write

$$\begin{aligned}
\int_x^y (t-x)^{2\alpha_1-1}(y-t)^{2\alpha_2-1}dt &= \int_0^{y-x} u^{2\alpha_1-1}(y-x-u)^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_2-1}\int_0^{y-x} u^{2\alpha_1-1}(1-\frac{u}{y-x})^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^{y-x} v^{2\alpha_1-1}(1-v)^{2\alpha_2-1}dv
\end{aligned}$$

and

$$\int_x^y (t-x)^{2\alpha_1-1}(y-t)^{2\alpha_2-1} \le (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^1 v^{2\alpha_1-1}(1-v)^{2\alpha_2-1}dv.$$

Finally we have

$$\begin{aligned}
\int_y^1 (t-x)^{2\alpha_1-1}(t-y)^{2\alpha_2-1}dt &= \int_0^{1-y} (u+y-x)^{2\alpha_1-1}u^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_1-1}\int_0^{1-y} \left(\frac{u}{y-x}+1\right)^{2\alpha_1-1} u^{2\alpha_2-1}du \\
&= (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^{\frac{1-y}{y-x}} (v+1)^{2\alpha_1-1}v^{2\alpha_2-1}dv
\end{aligned}$$

and

$$\int_y^1 (t-x)^{2\alpha_1-1}(t-y)^{2\alpha_2-1}dt \le (y-x)^{2\alpha_1+2\alpha_2-1}\int_0^{+\infty} (v+1)^{2\alpha_1-1}v^{2\alpha_2-1}dv.$$

thus it implies that

$$\int_0^1 |x-u|^{2\alpha_1-1}|y-u|^{2\alpha_2-1}du \le H_{\alpha_1\alpha_2}|y-x|^{2\alpha_1+2\alpha_2-1}$$

with

$$H_{\alpha_1\alpha_2} = \mathbf{B}(-2\alpha_1,-2\alpha_2) + \mathbf{B}(2\alpha_1, 3-2\alpha_1-2\alpha_2) + \mathbf{B}(2\alpha_2, 3-2\alpha_1-2\alpha_2).$$

To obtain the lower bound we write,

$$\int_0^x (x-u)^{2\alpha_1-1}(y-u)^{2\alpha_2-1}du \geq \int_0^x (y-u)^{2\alpha_1+2\alpha_2-2}du$$

that is also

$$\int_0^x (x-u)^{2\alpha_1-1}(y-u)^{2\alpha_2-1}du \geq \frac{y^{2\alpha_1+2\alpha_2-1}-(y-x)^{2\alpha_1+2\alpha_2-1}}{2\alpha_1+2\alpha_2-1}$$

and

$$\int_0^x (x-u)^{2\alpha_1-1}(y-u)^{2\alpha_2-1}du \geq 0.$$

Likewise we have

$$\int_y^1 (u-x)^{2\alpha_1-1}(y-u)^{2\alpha_2-1}du \geq 0.$$

Since we have also

$$\int_x^y (u-x)^{2\alpha_1-1}(y-u)^{2\alpha_2-1}du \geq (y-x)^{2\alpha_1-2\alpha_2-1}$$

we can conclude that

$$\int_0^1 |x-u|^{2\alpha_1-1}|y-u|^{2\alpha_2-1}du \geq |y-x|^{2\alpha_1+2\alpha_2-1}.$$

4.2 Proof of Theorem 4

Taking into account that

$$\int_0^1 |x-t|^{2\alpha_1-1}|y-t|^{2\alpha_2-1}dt \leq H_{\alpha_1\alpha_2}|x-y|^{2\alpha_1+2\alpha_2-1}$$

we get $\|K_{\alpha_1,\alpha_2}\| \leq \|K_{\alpha_1+\alpha_2}\|$ where $K_{\alpha_1+\alpha_2}$ is the integral operator on $L^2(0,1)$ with kernel $(x,y) \to |x-y|^{2\alpha_1+2\alpha_2-1}$ (see the demonstration of $\|K_{\alpha_1,\alpha_2,N}\|$ goes to zero in the proof of Lemma 5). Using the following proposition (see [8])

Proposition 1 *If $f = |\chi-\chi_0|^{-2\alpha}c$ with $c \in L^\infty(\mathbb{T})$ continuous and nonzero at $\chi_0 \in \mathbb{T}$ and $\alpha \in]0,\frac{1}{2}[$, if K_α is the integral operator on $L^2(0,1)$ with kernel $(x,y) \to |x-y|^{2\alpha-1}$ then we have*

$$\|T_N(f)\| \sim N^{2\alpha}C_\alpha\|K_\alpha\|c(\chi_0)$$

and

$$\psi(\alpha) \leq \|K_\alpha\| \leq \frac{1}{\alpha}$$

we obtain the upper bound for $\|K_{\alpha_1+\alpha_2}\|$.
Let $\mathbf{1}$ be the function which is identically 1 on $[0,1]$. We have, from Lemma 1

$$\|K_{\alpha_1,\alpha_2}\| \geq \frac{\|K_{\alpha_1,\alpha_2}\mathbf{1}\|}{\|\mathbf{1}\|} = \|K_{\alpha_1,\alpha_2}\mathbf{1}\| \geq \|K_{\alpha_1+\alpha_2}\mathbf{1}\|.$$

Since $K_{\alpha_1+\alpha_2}\mathbf{1}(1)(x) = \frac{1}{2(\alpha_1+\alpha_2)}\left(x^{2(\alpha_1+\alpha_2)}+(1-x)^{2(\alpha_1+\alpha_2)}\right)$, we obtain that $\|K_{\alpha_1,\alpha_2}\|$ is greater than or equal to

$$\frac{1}{4(\alpha_1+\alpha_2)}\int_0^1 \left(x^{2(\alpha_1+\alpha_2)}+(1-x)^{2(\alpha_1+\alpha_2)}\right)^2 dx = \psi(\alpha_1,\alpha_2).$$

This prove the lower bound for $\|K_{\alpha_1,\alpha_2}\|$.

5 Demonstration of Theorem 5

5.1 Demonstration of Lemma 2

5.1.1 First step : one singularity

Put $\sigma_\alpha = |1-\chi|^{-2\alpha}$ and $\sigma = |1-\chi|^{-2\alpha} c$ with $c \in A(r, \mathbb{T})$, where r will be precise later. First we prove $\hat{\sigma}(M) = |M^{2\alpha-1}| c(1)\,(1+o(1))$ uniformly in M. Of course we have for all $M \in \mathbb{Z}$ $\hat{\sigma}(M) = \sum_{u+v=M} \widehat{\sigma_\alpha}(u)\hat{c}(v)$. Let $\epsilon > 0$ and an integer $S_0 > 0$ such that

$$\forall S |S| \geq S_0 \qquad \sum_{|s| \leq S_0} \hat{c}(s) = c(1) + R_S \quad \text{and} \quad \widehat{\sigma_\alpha}(S) = C_\alpha |S|^{-2\alpha-1}(1 + r_S)$$

with $|R_S| \leq \epsilon$ and $|r_S| \leq \epsilon$. We have

$$\begin{aligned} \hat{\sigma}(M) &= \sum_{v<-S_0} \widehat{\sigma_\alpha}(M-v)\hat{c}(v) + \sum_{S_0 \geq v \geq -S_0} \widehat{\sigma_\alpha}(M-v)\hat{c}(v) \\ &+ \sum_{v>S_0} \widehat{\sigma_\alpha}(M-v)\hat{c}(v) \end{aligned}$$

Obviously

$$\Big| \sum_{v<-S_0} \sigma_\alpha(M-v)\hat{c}(v) \Big| \leq \max_{w \in \mathbb{Z}} |\widehat{\sigma}_\alpha(w)| \sum_{v<-S_0} |\hat{c}(v)|$$

and if $c \in A(r, \mathbb{T})$ and $S_0 = N^\nu$ $0 < \nu < 1$ we can conclude

$$\Big| \sum_{v<-S_0} \sigma_\alpha(M-v)\hat{c}(v) \Big| = O(N^{-r\nu}).$$

Now if ν is such that $-r\nu < 2\alpha - 1$ we obtain

$$\Big| \sum_{v<-S_0} \sigma_\alpha(M-v)\hat{c}(v) \Big| = o(N^{2\alpha-1}).$$

To have $r\nu < 2\alpha + 1$ with $\nu \in]0,1[$ and $\alpha \in]0, \frac{1}{2}[$ we must choose r in $]0,1]$. Moreover if $\alpha \in]-\frac{1}{2}, 0[$ we must pick α in $[2, +\infty[$. Clearly we have also

$$\Big| \sum_{v>S_0} \widehat{\sigma_\alpha}(M-v)\hat{c}(v) \Big| = o(N^{2\alpha-1}).$$

Moreover we have, if $|M| \geq 2S_0$,

$$\sum_{S_0 \geq v \geq -S_0} \widehat{\sigma_\alpha}(M-v)\hat{c}(v) = C_\alpha |M|^{2\alpha-1} c(1)\,(1+o(1))$$

that is the announced result.

5.1.2 Second step : two singularities

With the same notations than previously we can consider the Fourier coefficients of the function $\sigma = \sigma_{\alpha_2}(\chi_0\chi)\sigma_{\alpha_1}c$ with $\alpha_1 < \alpha_2$ and $\chi_0 \neq 1$. Following the first step we can assume $c = 1$ without lost of generality. For all $M \in \mathbb{Z}$ we have

$$\widehat{\sigma_{\alpha_2}(\chi_0\chi)} = \widehat{|1 - \chi_0\chi|^{\alpha_2}}(M) = \chi_0^{-M}\widehat{\sigma_{\alpha_2}}(M).$$

Let $\epsilon > 0$ and $S_0 > 0$ such that $S > S_0$ implies

- $$\sum_{-S \leq v \leq S} \widehat{\sigma_{\alpha_1}}(v)(\chi_0^{-v}) = \sigma_{\alpha_1}(\chi_0^{-1})(1 + R_1)$$
  with $|R_1| < \epsilon$.
- $$\sum_{-S \leq v \leq S} \widehat{\sigma_{\alpha_2}}(v)(\chi_0^{-v}) = \sigma_{\alpha_2}(\chi_0^{-1})(1 + R_2)$$
  with $|R_2| < \epsilon$.
- For all S such that $|S| > S_0$ we have
  $$\widehat{\sigma_1}(S) = C_{\alpha_1}|S|^{-2\alpha_1 - 1}(1 + R_{1,S})$$
  with $R_{1,S} = O(\epsilon)$.
- For all S such that $|S| > S_0$ we have
  $$\widehat{\sigma_2}(S) = C_{\alpha_2}|S|^{-2\alpha_2 - 1}(1 + R_{2,S})$$
  with $R_{2,S} = O(\epsilon)$.

Since

$$\hat{\sigma}(M) = \sum_{v \in \mathbb{Z}} \widehat{\sigma_{\alpha_1}}(M - v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v)$$

and

$$\hat{\sigma}(-M) = \sum_{v \in \mathbb{Z}} \widehat{\sigma_{\alpha_1}}(M - v)\chi_0^{v}\widehat{\sigma_{\alpha_2}}(v)$$

we can assume, without loss of generality, that $M > 0$. The aim of the rest of this demonstration is to prove that for M sufficiently large we have the formula

$$\sigma(M) = C_{\alpha_1}|M|^{2\alpha_1 - 1}c(1)\prod_{j=2}^{n}|\chi_0 - \chi|^{-2\alpha_j}(1 + R_M)$$

with $|R_M| = O(\epsilon)$.
Let ν be a fixed real such $0 < \nu < 1$. We write

$$\hat{\sigma}(M) = \sum_{i=0}^{5} \Sigma_i(M).$$

where

$$\Sigma_1(M) = \sum_{v \geq M + M^\nu} \widehat{\sigma_{\alpha_1}}(M - v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v) \quad \Sigma_2(M) = \sum_{M - M^\nu < v < M + M^\nu} \widehat{\sigma_{\alpha_1}}(M - v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v)$$

$$\Sigma_3(M)=\sum_{M^\nu\le v\le M-M^\nu}\widehat{\sigma_{\alpha_1}}(M-v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v)\quad \Sigma_4(M)=\sum_{-M^\nu<v\le M^\nu}\widehat{\sigma_{\alpha_1}}(M-v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v)$$

$$\Sigma_5(M)=\sum_{v\le -M^\nu}\widehat{\sigma_{\alpha_1}}(M-v)\chi_0^{-v}\widehat{\sigma_{\alpha_2}}(v).$$

Assume now $|M^\nu|>S_0$. We have

$$\Sigma_1(M)=C_{\alpha_1}C_{\alpha_2}\sum_{v\ge M+M^\nu}(v-M)^{2\alpha_1-1}v^{2\alpha_2-1}\chi_0^{-v}\left(1+R_1(M)\right)$$

with $R_1(M)=O(\epsilon)$. An Abel summation provides

$$\begin{aligned}\sum_{v\ge M+M^\nu}(v-M)^{2\alpha_1-1}v^{2\alpha_2-1}\chi_0^{-v}=&\sum_{v\ge M+M^\nu}\Big((v-M)^{2\alpha_1-1}v^{2\alpha_2-1}\\ -&\quad(v+1-M)^{2\alpha_1-1}(v+1)^{2\alpha_2-1}\Big)\tau_v+(M^\nu)^{2\alpha_1-1}\left(M+M^\nu\right)^{2\alpha_2-1}\tau_{S_0(M)-1}\end{aligned}$$

with $\tau_w=\sum_{h=1}^{w}\chi_0^{-h}$. For each $v\ge M+M^\nu$ the main value theorem gives us a real c_v $v<c_v<v+1$ such that

$$\begin{aligned}&\left((v-M)^{2\alpha_1-1}v^{2\alpha_2-1}-(v+1-M)^{2\alpha_1-1}(v+1)^{2\alpha_2-1}\right)=\\ =&\quad(-c_v-M)^{2\alpha_1-2}c_v^{2\alpha_2-2}\left((c_v-M)(2\alpha_2+1)+c_v(2\alpha_1+1)\right)\end{aligned}$$

from this equality we infer

$$\left((v-M)^{2\alpha_1-1}v^{2\alpha_2-1}-(v+1-M)^{2\alpha_1-1}(v+1)^{2\alpha_2-1}\right)=O\left((v-M)^{2\alpha_1-2}v^{2\alpha_2-2}\right)$$

and

$$\begin{aligned}&\Big|\sum_{v\ge M+M^\nu}\left(\left((v-M)^{2\alpha_1-1}v^{2\alpha_2-1}-(v+1-M)^{2\alpha_1-1}(v+1)^{2\alpha_2-1}\right)\tau_v\right)\Big|=\\ =&\sum_{v\ge M+M^\nu}O\left((v-M)^{2\alpha_1-2}v^{2\alpha_2-2}\right)=O\left((M+S_0(M))^{2\alpha_2-1}\right)=o(M^{2\alpha_1-1})\end{aligned}$$

Since

$$\left|(M^\nu)^{2\alpha_1-1}\left(M+M^\nu\right)^{2\alpha_2-1}\tau_{S_0(M)-1}\right|=o(M^{2\alpha_1-1})$$

we have $\Sigma_1(M)=o(M^{2\alpha_1-1})$, and $\Sigma_1(M)=O(\epsilon M^{2\alpha_1-1})$ for a sufficiently large M. The bounds $M>M^\nu>S_0$ implies

$$\Sigma_2(M)=M^{2\alpha_2-1}C_{\alpha_2}|1-\chi_0|^{2\alpha_1}\chi_0^{-M}\left(1+R_2(M)\right)$$

with $R_2(M)=O(\epsilon)$). Then $-\alpha_2<-\alpha_1$ provides $\Sigma_2(M)=o(M^{2\alpha_1-1})$. The hypothesis on M gives us

$$\Sigma_3(M)=C_{\alpha_1}C_{\alpha_2}\sum_{v=M^\nu}^{M-M^\nu}(M-v)^{2\alpha_1-1}v^{2\alpha_2-1}\chi_0^{-v}\left(1+R_3(M)\right)$$

with $R_3(M)=O(\epsilon)$. Always with an Abel summation we obtain

$$\sum_{v=M^\nu}^{M-M^\nu}(M-v)^{2\alpha_1-1}v^{2\alpha_2-1}\chi_0^{-v}=A_1+A_2$$

with

$$A_1 = C_{\alpha_1} C_{\alpha_2} \sum_{v=S_0(M)}^{M-S_0(M)} \left((M-v)^{-2\alpha_1 -1} v^{2\alpha_2 -1} - (M-v-1)^{2\alpha_1 -1} (v+1)^{2\alpha_2 -1} \right) \tau_v$$

and

$$A_2 = \tau_{S_0(M)-1} (M - S_0(M))^{2\alpha_1 -1} \left(S_0(M)\right)^{2\alpha_2 -1} - \tau_{M-S_0(M)-1} (M - S_0(M))^{2\alpha_2 -1} \left(S_0(M)\right)^{2\alpha_1 -1}.$$

As previously for each integer v such that $M^\nu \leq v \leq M - M^\nu$ we have a real c_v $v < c_v < v+1$ such that

$$\begin{aligned} &\left((M-v)^{2\alpha_1 -1} v^{2\alpha_2 -1} - (M-v-1)^{2\alpha_1 -1} (v+1)^{2\alpha_2 -1} \right) = \\ = \quad & O(c_v^{2\alpha_2 -2} (M - c_v)^{2\alpha_1 -2}) \leq O\left((v(M-v))^{2\alpha_1 -2} \right). \end{aligned}$$

The study of the function $x \to x(M-x)$ on $[M^\nu, M - M^\nu]$ gives

$$\left((M-v)^{2\alpha_1 -1} v^{2\alpha_2 -1} - (M-v-1)^{2\alpha_1 -1} (v+1)^{2\alpha_2 -1} \right) \leq O(M^{2\alpha_1 -2}) = o(M^{2\alpha_1 -1}).$$

Moreover it is easily seen that

$$A_2 = o(M^{-2\alpha_1 -1}).$$

Hence for sufficiently large M we may write $\Sigma_3(M) = O(M^{2\alpha_1 -1})$. We obtain also

$$\Sigma_4(M) = C_{\alpha_1} M^{2\alpha_1 -1} |1 - \chi_0|^{2\alpha_2} \left(1 + o(1)\right)$$

and as for Σ_1 we have $\Sigma_5(M) = o(M^{2\alpha_1 -1})$. Finally we have obtained the asymptotic expansion

$$\forall M \quad \text{such} \quad \text{that} \quad M^\nu > S_0 \quad \hat{\sigma}(M) = C_{\alpha_1} M^{2\alpha_1 -1} |1 - \chi_0|^{-2\alpha_2} \left(1 + O(\epsilon)\right),$$

that was the aim of our demonstration.

5.1.3 n and $n+1$ singularities.

Let $\sigma = |1 - \chi|^{-2\alpha_1} | \prod_{j=2}^{n} |\chi - \chi_0|^{-2\alpha_j} c$ with $c \in A(r, \mathbb{T})$ $0 < r < 1$ and $-\alpha_1 > -\alpha_j, \quad \forall j, \quad 2 \leq j \leq n$. Assume that for $\epsilon > 0$ and a sufficiently large M we have

$$\hat{\sigma}(M) = C_{\alpha_1} |M|^{2\alpha_1 -1} c(1) \prod_{j=2}^{n} |\chi_0 - 1|^{-2\alpha_j} (1 + R_M)$$

with $|R_M| \leq \epsilon$. If $\sigma' = |1 - \chi|^{-2\alpha_1} \prod_{j=2}^{n+1} |\chi - \chi_0|^{-2\alpha_j} c$ $c \in A(r, \mathbb{T})$ $0 < r < 1$ and $\alpha_1 > \alpha_j, \quad \forall j, \quad 2 \leq j \leq n+1$, we prove exactly as for the precedent point that σ' has the same property that σ, that ends the proof of the present lemma.

5.2 Proof of Theorem 5 and Corollary 2

The proof is the same than for the theorem 3. We can write $T_N(\tilde{f}_1) = \tilde{T}_{1,N} + \tilde{D}_{1,N}$ and $T_N(\tilde{f}_2) = \tilde{T}_{2,N} + \tilde{D}_{2,N}$, with if $k \neq l$

$$\left(\tilde{T}_{1,N}\right)_{k+1,l+1} = c_1(1)C_\alpha |l-k|^{2\alpha-1} \prod_{j=1}^{p} |1-\chi_j|^{-2\alpha_j}$$

$$\left(\tilde{T}_{2,N}\right)_{k+1,l+1} = c_2(1)C_\beta |l-k|^{2\beta-1} \prod_{j=1}^{q} |1-\chi_j|^{-2\beta_j}$$

and
$\left(\tilde{T}_{1,N}\right)_{k+1,k+1} = 0$, $\left(\tilde{T}_{2,N}\right)_{k+1,k+1} = 0$. Then $\tilde{D}_{1,N}$ and $\tilde{D}_{2,N}$ are Toeplitz matrices with $(\tilde{D}_{1,N})_{k+1,l+1} = o|k-l|^{2\alpha-1}$ and $(\tilde{D}_{2,N})_{k+1,l+1} = o|k-l|^{2\beta-1}$. hence we have (see [8]) $\|\tilde{D}_{1,N}\| = o(N^{2\alpha})$ and $\|\tilde{D}_{2,N}\| = o(N^{2\beta})$. As for the demonstration of Theorem 3 we have

$$\begin{aligned} \|T_N(\tilde{f}_1)T_N(\tilde{f}_2)\| &= \|\tilde{T}_{1,N}\tilde{T}_{2,N}\| + o(N^{2\alpha 2\beta}) \\ &= CN^{2\alpha 2\beta}\|K_{\alpha,\beta}\| + o(N^{2\alpha+2\beta}) \end{aligned}$$

with

$$C = c_1(1)c_2(1)C_\alpha C_\beta \prod_{j=1}^{p} |1-\chi_j|^{-2\alpha_j} \prod_{j=1}^{q} |1-\chi_j|^{-2\beta_j}.$$

Corollary 2 is a direct consequence of the equality

$$T_N(|\chi_0-\chi|^{-2\alpha}\psi_1) = \Delta_0(\chi_0)T_N\left(|1-\chi|^{-2\alpha}\psi_{1,\chi_0}\right)\Delta_0^{-1}(\chi_0)$$

and

$$T_N(|\chi_0-\chi|^{-2\beta}\psi_2) = \Delta_0(\chi_0)T_N\left(|1-\chi|^{-2\beta}\psi_{2,\chi_0}\right)\Delta_0^{-1}(\chi_0)$$

where $\Delta_0(\chi_0)$ is as in the introduction and

$$\psi_1 = \prod_{j=1}^{p} |\chi_j - \chi|^{-2\alpha_j} c_1$$

$$\psi_2 = \prod_{j=1}^{q} |\chi_j - \chi|^{-2\beta_j} c_2$$

and

$$\psi_{1,\chi_0}(\chi) = \psi_1(\chi_0\chi) \quad \text{and} \quad \psi_{2,\chi_0}(\chi) = \psi_2(\chi_0\chi).$$

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